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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ, ЧИСЛЕННЫЕ МЕТОДЫ И КОМПЛЕКСЫ ПРОГРАММ/MATHEMATICAL MODELING, NUMERICAL METHODS AND PROGRAM COMPLEXES

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STRUCTURAL PROPERTIES OF DISTRIBUTIVE HESITANT FUZZY $\mathscr{R}$-MODULES

Research article

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Abstract

In this study, we investigate distributive hesitant fuzzy $\mathscr{R}$-modules and explore some of their main structural properties within hesitant fuzzy algebraic systems. We focus on the behavior of distributivity and its interaction with module-theoretic conditions, establishing new results that clarify its algebraic role and structural significance. We prove that every chained hesitant fuzzy $\mathscr{R}$-module is distributive, whereas the converse is not true in general; thus, distributivity does not impose chain conditions on hesitant fuzzy $\mathscr{R}$-modules. Moreover, distributivity can be effectively characterized by level submodules, providing a useful tool for analyzing these structures and revealing a clear connection between distributive modules and their level submodules. These findings extend recent developments in hesitant fuzzy module theory and contribute to a deeper understanding of distributivity. They also provide a stronger theoretical basis for future research on hesitant fuzzy $\mathscr{R}$-modules and related areas of fuzzy algebra.

Keywords: distributive module, hesitant fuzzy module, direct sum, level submodules, homomorphism.

СТРУКТУРНЫЕ СВОЙСТВА ДИСТРИБУТИВНЫХ НЕРЕШИТЕЛЬНЫХ НЕЧЁТКИХ $\mathscr{R}$-МОДУЛЕЙ

Научная статья

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Аннотация

В данном исследовании рассматриваются дистрибутивные нерешительные нечёткие $\mathscr{R}$-модули и изучаются некоторые их основные структурные свойства в рамках нерешительных нечётких алгебраических систем. Основное внимание уделяется поведению дистрибутивности и её взаимодействию с теоретико-модульными условиями; устанавливаются новые результаты, уточняющие её алгебраическую роль и структурное значение. Доказано, что каждый цепной нерешительный нечёткий $\mathscr{R}$-модуль является дистрибутивным, тогда как обратное утверждение в общем случае неверно; таким образом, дистрибутивность не накладывает условий цепей на нерешительные нечёткие $\mathscr{R}$-модули. Кроме того, дистрибутивность может быть эффективно охарактеризована с помощью уровневых подмодулей, что предоставляет полезный инструмент для анализа данных структур и выявляет чёткую связь между дистрибутивными модулями и их уровневыми подмодулями. Эти результаты расширяют недавние достижения в теории нерешительных нечётких модулей и способствуют более глубокому пониманию дистрибутивности. Они также создают более прочную теоретическую основу для будущих исследований нерешительных нечётких $\mathscr{R}$-модулей и смежных областей нечёткой алгебры.

Ключевые слова: дистрибутивный модуль, нерешительный нечёткий модуль, прямая сумма, уровневые подмодули, гомоморфизм.

Introduction

The fuzzy concept of a Distributive Module was introduced as a natural extension of classical Distributive Module by adding fuzziness to the submodules involved in the operation of distributive laws using fuzzy intersection and sum operations [1]. This approach allows for a more flexible representation of the algebraic structures, especially when dealing with partial membership and uncertainty. As an extension of conventional fuzzy sets, hesitant fuzzy sets can represent the grade of membership of an element by a finite set of possible values instead of only one value. This paradigm has proven to be very effective in modelling uncertainty in algebraic and decision-making settings. In this respect, the hesitant fuzzy algebraic structures in general and in particular, hesitant fuzzy $\mathscr{R}$-modules have received much attention from the research community in recent years. Along these lines, new families of hesitant fuzzy algebraic structures continue to be introduced and studied [10].

Various fundamental properties of hesitant fuzzy $\mathscr{R}$-modules have been studied, including homomorphisms, submodule configurations and lattice-theoretic properties. In particular, chained hesitant fuzzy $\mathscr{R}$-modules have been studied due to their strong structural properties, in which the set of hesitant fuzzy submodules is totally ordered with inclusion [2], [3]. This is especially helpful when configuring the inside of such modules.

On the other hand, in [4] the distributive hesitant fuzzy $\mathscr{R}$-modules were studied where the relationship with classical distributive modules and fuzzy distributive modules was built. These investigations show that level submodules can often be

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used to describe distributivity, and thus establish a connection between classical module theory and hesitant fuzzy paradigms. Recent advances in fuzzy algebra show that level-based characterizations are very powerful tools for transferring structural properties from classical to fuzzy environments [5], [6]. This viewpoint is also supported by recent work outside module theory: the categorical and functorial aspects of fuzzy modules have been clarified [11], while the interaction between fuzzy sets and lattice-valued (residuated) structures has been examined in detail [12], which together suggest that the level-based approach adopted here fits naturally into current research. Moreover, in classical module theory the relations between distributivity and other structural properties like chain properties, noetherian properties and direct sum decompositions have been studied and have been extended into a fuzzy environment to a certain degree [7], [8], [9].

The aim of this paper is to study distributive hesitant fuzzy $\mathscr{R}$-modules by means of level submodule techniques, without introducing any new definitions. The principal goal is to develop new structural theorems relating distributivity and the chained property. The preservation of distributivity under isomorphisms, homomorphisms and direct sums with appropriate annihilator conditions is studied in particular. Furthermore, several level-based characterizations are derived in order to elucidate the relationship between distributive and chained hesitant fuzzy $\mathscr{R}$-modules.

The main new contributions of the paper are the following:

• the invariance of distributivity of hesitant fuzzy $\mathscr{R}$-modules under $\mathscr{R}$-module isomorphisms (Proposition 2.2);

• a criterion for the distributivity of hesitant fuzzy direct sums under the annihilator condition $\text{ann}(\mathscr{M}_1) + \text{ann}(\mathscr{M}_2) = \mathscr{R}$, together with an example showing that this condition cannot be omitted (Theorem 2.4, Example 2.5);

• a level-wise description of the behaviour of images and inverse images of hesitant fuzzy submodules of distributive hesitant fuzzy $\mathscr{R}$-modules under homomorphisms (Proposition 2.6);

• a proof that every chained hesitant fuzzy $\mathscr{R}$-module is distributive, whereas the converse fails in general (Theorem 2.9, Remark 2.10). Although the corresponding classical statements are well known, the results listed above are obtained entirely within the hesitant fuzzy framework, where the level-set technique is used to transfer the arguments between the classical and the hesitant fuzzy settings.

1.1. Preliminaries and Basic Definitions

In this subsection some of the basic definitions, concepts and preliminary results needed for the subsequent development of this paper are presented. The materials contain the theoretical underpinning for the arguments and framework of the study.

Definition 1.1.1. [1] Let S be a nonempty set. A fuzzy set $\mathcal{G}$ on S assigns to every element of S a membership value lying in the interval [0, 1]. In symbolic form, this is written as $\mathcal{G} : S \to [0, 1]$.

Definition 1.1.2. [5] For a fuzzy set $\mathcal{G}$ in S and each $t \in [0, 1]$, the set $\mathcal{G}_t = \{x \in S \mid \mathcal{G}(x) \geq t\}$, is termed the level subset of $\mathcal{G}$.

Definition 1.1.3. [2] Consider an $\mathscr{R}$-module $\mathscr{M}$. A fuzzy set $\mathscr{X}$ on $\mathscr{M}$ is designated a fuzzy module if for all $x, y \in \mathscr{M}$ and $r \in \mathscr{R}$:

- $\mathcal{X}(x - y) \geq \min\{\mathcal{X}(x), \mathcal{X}(y)\}$;
- $\mathcal{X}(rx) \geq \mathcal{X}(x)$;
- $\mathcal{X}(0) = 1$.

Definition 1.1.4. [3] Let $\mathcal{X}$ be a fuzzy module on $\mathscr{M}$. A fuzzy set $\mathcal{G}$ is termed a fuzzy submodule of $\mathcal{X}$ if $\mathcal{G}(x) \leq \mathcal{X}(x)$, $\forall x \in \mathscr{M}$.

Remark 1.1.5. [5] A fuzzy set $\mathcal{G}$ constitutes a fuzzy submodule of $\mathscr{X}$ precisely when $\mathcal{G}_t$ forms a submodule of $\mathscr{M}$ for every $t \in (0, 1]$.

Definition 1.1.6. [1], [2], [3], [4] Let $\mathscr{X}$ be a fuzzy module over an $\mathscr{R}$-module $\mathscr{M}$. Then $\mathscr{X}$ is designated a fuzzy distributive module if for all fuzzy submodules $\mathcal{G}, \mathcal{H}, \mathcal{I}$ of $\mathscr{X}$:

$$\mathcal{G} \cap (\mathcal{H} + \mathcal{J}) = (\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{J}).$$

Remark 1.1.7. [1], [4] A fuzzy module $\mathscr{X}$ exhibits distributivity exactly when $\mathscr{X}_t$ constitutes a distributive module for every $t \in (0, 1]$.

Definition 1.1.8. [1] Consider fuzzy $\mathscr{R}$-modules $\mathscr{X}$ and $\mathscr{Y}$ on $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. The fuzzy direct sum is defined by:

$$(\mathcal{X} \oplus \mathcal{Y})(x, y) = \min\{\mathcal{X}(x), \mathcal{Y}(y)\}.$$

Definition 1.1.9. [2], [3] Let $\mathscr{F}$ be a hesitant fuzzy $\mathscr{R}$-module on an $\mathscr{R}$-module $\mathscr{M}$. Then $\mathscr{F}$ is termed a chained hesitant fuzzy $\mathscr{R}$-module if for any hesitant fuzzy submodules $\mathcal{G}$ and $\mathcal{H}$ of $\mathscr{F}$, either $\mathcal{G} \subseteq \mathcal{H}$ or $\mathcal{H} \subseteq \mathcal{G}$.

These foundational concepts establish the groundwork for examining distributive structures in hesitant fuzzy $\mathscr{R}$-modules. Throughout the paper a unified system of notation is adopted: $\mathscr{X}$ and $\mathscr{Y}$ denote fuzzy modules; $\mathscr{F}$, $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{K}$ denote hesitant fuzzy $\mathscr{R}$-modules; $\mathcal{G}, \mathcal{H}, \mathcal{I}$ (possibly with indices) denote fuzzy or hesitant fuzzy submodules; and $\mathcal{G}_j$ denotes the level subset of $\mathcal{G}$ at a level $j \in (0, 1]$. For the basic notions of hesitant fuzzy sets, hesitant fuzzy $\mathscr{R}$-modules and their level subsets we refer to [2], [3].

Structural Properties of Distributive Hesitant Fuzzy $\mathscr{R}$-Modules

This section develops novel findings on distributive hesitant fuzzy $\mathscr{R}$-modules by extending classical fuzzy distributive structures to the hesitant fuzzy framework. The mathematical notation $\mathscr{R}$ denotes a commutative ring possessing an identity element while $\mathscr{F}$ represents a hesitant fuzzy $\mathscr{R}$-module that operates on the $\mathscr{R}$-module $\mathscr{M}$.

Lemma 2.1. Let $g : \mathscr{M} \to \mathscr{N}$ be an $\mathscr{R}$-module isomorphism, let $\mathscr{F}$ be a hesitant fuzzy $\mathscr{R}$-module on $\mathscr{M}$ and let $\mathcal{G}$, $\mathcal{H}$ be hesitant fuzzy submodules of $\mathscr{F}$. Then $g(\mathcal{G} \cap \mathcal{H}) = g(\mathcal{G}) \cap g(\mathcal{H})$ and $g(\mathcal{G} + \mathcal{H}) = g(\mathcal{G}) + g(\mathcal{H})$.

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Proof. For every $j \in (0, 1]$ we have $(g(\mathcal{G}))_j = g\left(\mathcal{G}_j\right)$ and, since g is bijective, $g\left(\mathcal{G}_j \cap \mathcal{H}_j\right) = g\left(\mathcal{G}_j\right) \cap g\left(\mathcal{H}_j\right)$ and $g\left(\mathcal{G}_j + \mathcal{H}_j\right) = g\left(\mathcal{G}_j\right) + g\left(\mathcal{H}_j\right)$. Passing to level subsets, we obtain $(g(\mathcal{G} \cap \mathcal{H}))_j = (g(\mathcal{G}) \cap g(\mathcal{H}))_j$ and $(g(\mathcal{G} + \mathcal{H}))_j = (g(\mathcal{G}) + g(\mathcal{H}))_j$ for all $j \in (0, 1]$, which proves the assertion.

Proposition 2.2. Distributivity of hesitant fuzzy $\mathcal{R}$-modules remains invariant under $\mathcal{R}$-module isomorphisms.

Proof. Suppose $g : \mathcal{M} \to \mathcal{N}$ defines an $\mathcal{R}$-module isomorphism. Define a hesitant fuzzy set $g(\mathcal{F})$ on $\mathcal{N}$ by $g(\mathcal{F})(y) = \mathcal{F}\left(g^{-1}(y)\right)$ for every $y \in \mathcal{N}$.

Let $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$ be hesitant fuzzy submodules of $\mathcal{F}$.

Since g constitutes an $\mathcal{R}$-module isomorphism, it follows from Lemma 1 that g preserves sums and intersections of hesitant fuzzy submodules.

Thus, we obtain:

$$g(\mathcal{G} \cap (\mathcal{H} + \mathcal{I})) = g(\mathcal{G}) \cap (g(\mathcal{H}) + g(\mathcal{I})),$$
$$g((\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I})) = (g(\mathcal{G}) \cap g(\mathcal{H})) + (g(\mathcal{G}) \cap g(\mathcal{I})).$$

This demonstrates that the equality $\mathcal{G} \cap (\mathcal{H} + \mathcal{I}) = (\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I})$ is equivalent to:

$$g(\mathcal{G}) \cap (g(\mathcal{H}) + g(\mathcal{I})) = (g(\mathcal{G}) \cap g(\mathcal{H})) + (g(\mathcal{G}) \cap g(\mathcal{I})).$$

Consequently, $\mathcal{F}$ exhibits distributivity if and only if $g(\mathcal{F})$ does as well.

Lemma 2.3. [1] Let $\mathcal{M}_1$, $\mathcal{M}_2$ be $\mathcal{R}$-modules satisfying the condition that the sum of the annihilators of $\mathcal{M}_1$ and $\mathcal{M}_2$ equals the entire ring $\mathcal{R}$. The direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ forms a distributive $\mathcal{R}$ module when both $\mathcal{M}_1$ and $\mathcal{M}_2$ function as distributive $\mathcal{R}$-modules.

Theorem 2.4. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be distributive hesitant fuzzy $\mathcal{R}$-modules on $\mathcal{R}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. If $\text{ann}(\mathcal{M}_1) + \text{ann}(\mathcal{M}_2) = \mathcal{R}$, then the hesitant fuzzy direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ constitutes a distributive hesitant fuzzy $\mathcal{R}$-module on $\mathcal{M}_1 \oplus \mathcal{M}_2$.

Proof. For any level $j \in (0, 1]$, we have:

$$\left[\left(\mathcal{F}_1 \oplus \mathcal{F}_2\right)_j\right] = \left[\left(\mathcal{F}_1\right)_j \oplus \left(\mathcal{F}_2\right)_j\right].$$

Given that $\mathcal{F}_1$ and $\mathcal{F}_2$ are distributive hesitant fuzzy modules, $(\mathcal{F}_1)_j$ and $(\mathcal{F}_2)_j$ are distributive $\mathcal{R}$-modules for all j. By the classical result concerning direct sums of distributive modules under the annihilator condition (as per Lemma 2.3), $(\mathcal{F}_1)_j \oplus (\mathcal{F}_2)_j$ exhibits distributivity. Hence, $(\mathcal{F}_1 \oplus \mathcal{F}_2)_j$ is distributive for all j, and consequently $\mathcal{F}_1 \oplus \mathcal{F}_2$ qualifies as a distributive hesitant fuzzy $\mathcal{R}$-module.

Example 2.5. Take $\mathcal{R} = \mathbb{Z}$, with $\mathcal{M} = \mathbb{Z} \oplus \mathbb{Z}$. Define $\mathcal{F}(x) = \{1\}$ for all $x \in \mathcal{M}$. Consider the hesitant fuzzy submodules corresponding to $\mathcal{G} = \mathbb{Z}(1, 0)$, $\mathcal{H} = \mathbb{Z}(0, 1)$, $\mathcal{I} = \mathbb{Z}(1, 1)$. Then $\mathcal{H} + \mathcal{I} = \mathcal{M}$, so $\mathcal{G} \cap (\mathcal{H} + \mathcal{I}) = \mathcal{G} \neq 0$. On the other hand, $\mathcal{G} \cap \mathcal{H} = 0$ and $\mathcal{G} \cap \mathcal{I} = 0$, so that $(\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I}) = 0$. Hence $\mathcal{G} \cap (\mathcal{H} + \mathcal{I}) \neq (\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I})$, and therefore $\mathcal{F}$ is not a distributive hesitant fuzzy $\mathcal{R}$-module.

This example demonstrates that assigning complete hesitant membership to every element fails to guarantee fulfillment of the distributive property for hesitant fuzzy $\mathcal{R}$-modules. Note also that here $\text{ann}(\mathbb{Z}) + \text{ann}(\mathbb{Z}) = 0 \neq \mathbb{Z}$, so the example does not contradict Theorem 1; on the contrary, it shows that the annihilator condition of Theorem 2.4 cannot be omitted.

Proposition 2.6. Let $\mathcal{F}$ and $\mathcal{K}$ be hesitant fuzzy $\mathcal{R}$-modules on $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$, respectively, and let $f : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{R}$-homomorphism inducing a hesitant fuzzy homomorphism $f : \mathcal{F} \to \mathcal{K}$. If $\mathcal{F}$ is a distributive hesitant fuzzy $\mathcal{R}$-module, then the following results obtain:

1. For every hesitant fuzzy submodule $\mathcal{G}$ of $\mathcal{F}$, the image $f(\mathcal{G})$ constitutes a hesitant fuzzy submodule of $\mathcal{K}$.
2. For every hesitant fuzzy submodule $\mathcal{H}$ of $\mathcal{K}$, the inverse image $f^{-1}(\mathcal{H})$ forms a hesitant fuzzy submodule of the distributive hesitant fuzzy $\mathcal{R}$-module $\mathcal{F}$.
3. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\mathcal{F}$-invariant hesitant fuzzy submodules of $\mathcal{F}$, then $f\left(\mathcal{G}_1 \cap \mathcal{G}_2\right) = f\left(\mathcal{G}_1\right) \cap f\left(\mathcal{G}_2\right)$.
4. If f is a monomorphism and $\mathcal{H}_1$, $\mathcal{H}_2$ are hesitant fuzzy submodules of $\mathcal{K}$, then $f^{-1}\left(\mathcal{H}_1 \cap \mathcal{H}_2\right) = f^{-1}\left(\mathcal{H}_1\right) \cap f^{-1}\left(\mathcal{H}_2\right)$.
5. For any hesitant fuzzy submodules $\mathcal{G}_1$, $\mathcal{G}_2$ of $\mathcal{F}$, $f\left(\mathcal{G}_1 + \mathcal{G}_2\right) = f\left(\mathcal{G}_1\right) + f\left(\mathcal{G}_2\right)$.

Proof. We establish the assertions using the level-wise characterization of hesitant fuzzy submodules. Let $j \in (0, 1]$. Since $\mathcal{G}$ is a hesitant fuzzy submodule of $\mathcal{F}$, its level subset $\mathcal{G}_j$ forms a submodule of $\mathcal{F}_j$. By the definition of hesitant fuzzy images, we have $(f(\mathcal{G}))_j = f\left(\mathcal{G}_j\right)$, which constitutes a submodule of $\mathcal{K}_j$. Hence, $f(\mathcal{G})$ qualifies as a hesitant fuzzy submodule of $\mathcal{K}$. This establishes (1).

Similarly, for any hesitant fuzzy submodule $\mathcal{H}$ of $\mathcal{K}$, the level subset $(f^{-1}(\mathcal{H}))_j = f^{-1}(\mathcal{H}_j)$ forms a submodule of $\mathcal{F}_j$, demonstrating that $f^{-1}(\mathcal{H})$ is a hesitant fuzzy submodule of $\mathcal{F}$. This proves (2).

Assume now that $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\mathcal{F}$-invariant hesitant fuzzy submodules of $\mathcal{F}$. Then, for every $j \in (0, 1]$, $\left(\mathcal{G}_1\right)_i$ and $\left(\mathcal{G}_2\right)_i$ are $\mathcal{F}_j$-invariant submodules of $\mathcal{F}_j$. By the classical module result, $f\left(\left(\mathcal{G}_1\right)_j \cap \left(\mathcal{G}_2\right)_j\right) = f\left(\left(\mathcal{G}_1\right)_j\right) \cap f\left(\left(\mathcal{G}_2\right)_j\right)$. Employing level subsets, we obtain $\left(f\left(\mathcal{G}_1 \cap \mathcal{G}_2\right)\right)_j = \left(f\left(\mathcal{G}_1\right)\right)_j \cap \left(f\left(\mathcal{G}_2\right)\right)_j$, and therefore $f\left(\mathcal{G}_1 \cap \mathcal{G}_2\right) = f\left(\mathcal{G}_1\right) \cap f\left(\mathcal{G}_2\right)$. This verifies (3)

If f is injective, then for any hesitant fuzzy submodules $\mathcal{H}_1$, $\mathcal{H}_2$ of $\mathcal{K}$, we have for all $j \in (0, 1]$, $f^{-1}((\mathcal{H}_1)_j \cap (\mathcal{H}_2)_j) = f^{-1}((\mathcal{H}_1)_j) \cap f^{-1}((\mathcal{H}_2)_j)$, which implies $f^{-1}(\mathcal{H}_1 \cap \mathcal{H}_2) = f^{-1}(\mathcal{H}_1) \cap f^{-1}(\mathcal{H}_2)$, confirming (4).

Finally, for hesitant fuzzy submodules $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{F}$, we have:

$$\left(f\left(\mathcal{G}_1 + \mathcal{G}_2\right)\right)_j = f\left(\left(\mathcal{G}_1\right)_j + \left(\mathcal{G}_2\right)_j\right) = f\left(\left(\mathcal{G}_1\right)_j\right) + f\left(\left(\mathcal{G}_2\right)_j\right) = \left(f\left(\mathcal{G}_1\right) + f\left(\mathcal{G}_2\right)\right)_j, \text{ for all } j \in (0, 1],$$

which yields $f\left(\mathcal{G}_1 + \mathcal{G}_2\right) = f\left(\mathcal{G}_1\right) + f\left(\mathcal{G}_2\right)$.

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Remark 2.7. The $\mathscr{F}$-invariance condition proves essential to ensure preservation of intersections under the image mapping (see [1]).

Remark 2.8. Although the statements of Proposition 2.6 resemble the classical properties of images and inverse images of submodules under module homomorphisms, they cannot be deduced directly from the known results of the literature: a hesitant fuzzy submodule is determined by the whole family of its level subsets simultaneously, so the classical statements have to be verified level-wise for every $j \in (0, 1]$ and then lifted to the hesitant fuzzy setting. Proposition 2.6 makes this transfer explicit and, in items (3) and (4), specifies the additional assumptions ($\mathscr{F}$-invariance and injectivity of f) under which intersections are preserved in the hesitant fuzzy environment.

Theorem 2.9. Every chained hesitant fuzzy $\mathscr{R}$-module is a distributive hesitant fuzzy $\mathscr{R}$-module.

Proof. Let $\mathscr{F}$ be a chained hesitant fuzzy $\mathscr{R}$-module on $\mathscr{M}$ and let $\mathcal{G}$, $\mathscr{H}$, $\mathscr{I}$ be hesitant fuzzy submodules of $\mathscr{F}$. Since $\mathscr{F}$ is chained, its hesitant fuzzy submodules are totally ordered by inclusion, so either $\mathscr{H} \subseteq \mathscr{I}$ or $\mathscr{I} \subseteq \mathscr{H}$; assume, without loss of generality, that $\mathscr{H} \subseteq \mathscr{I}$. Then $\mathscr{H} + \mathscr{I} = \mathscr{I}$ and $\mathcal{G} \cap \mathscr{H} \subseteq \mathcal{G} \cap \mathscr{I}$, whence $(\mathcal{G} \cap \mathscr{H}) + (\mathcal{G} \cap \mathscr{I}) = \mathcal{G} \cap \mathscr{I}$. Consequently, $\mathcal{G} \cap (\mathscr{H} + \mathscr{I}) = \mathcal{G} \cap \mathscr{I} = (\mathcal{G} \cap \mathscr{H}) + (\mathcal{G} \cap \mathscr{I})$. Therefore, $\mathscr{F}$ is a distributive hesitant fuzzy $\mathscr{R}$-module.

Remark 2.10. The converse of Theorem 2.9 does not hold in general; consequently, distributivity is independent of chain conditions. Indeed, take $\mathscr{R} = \mathbb{Z}$ and $\mathscr{M} = \mathbb{Z}$, and define $\mathscr{F}(x) = \{1\}$ for all $x \in \mathbb{Z}$. For every $j \in (0, 1]$ the level subset $\mathscr{F}_j = \mathbb{Z}$ is a distributive $\mathbb{Z}$-module, since the lattice operations on the submodules $n\mathbb{Z}$ of $\mathbb{Z}$ are given by greatest common divisors and least common multiples, which distribute over each other. Hence $\mathscr{F}$ is a distributive hesitant fuzzy $\mathbb{Z}$-module. Nevertheless, $\mathscr{F}$ is not chained, because the hesitant fuzzy submodules corresponding to $2\mathbb{Z}$ and $3\mathbb{Z}$ are incomparable with respect to inclusion. Thus a distributive hesitant fuzzy $\mathscr{R}$-module need not satisfy any chain-type condition.

Conclusion

In this paper, distributive hesitant fuzzy $\mathscr{R}$-modules were studied through a structural and level-based approach. The obtained results extend several classical properties of distributive modules to the hesitant fuzzy setting and provide a clearer understanding of the interaction between distributivity and hesitant fuzzy submodule structures. It was shown that distributivity is preserved under $\mathscr{R}$-module isomorphisms and under hesitant fuzzy direct sums satisfying suitable annihilator conditions. Moreover, the behaviour of distributivity with respect to homomorphisms was analysed by level submodule methods. It was also proved that every chained hesitant fuzzy $\mathscr{R}$-module is distributive (Theorem 2.9), whereas the converse is not true in general (Remark 2.10); hence distributivity is independent of chain conditions in the internal structure of hesitant fuzzy modules. In general, the above results are helpful for the development of hesitant fuzzy module theory and can be used as a starting point for studying further the more general fuzzy algebraic structures and corresponding module-theoretic properties.

Конфликт интересов

Не указан.

Рецензия

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Conflict of Interest

None declared.

Review

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