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STRUCTURAL PROPERTIES OF DISTRIBUTIVE HESITANT FUZZY $\mathcal{R}$ -MODULES

Research article

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Abstract

This paper develops structural results for distributive hesitant fuzzy $\mathcal{R}$ -modules and separates the genuinely distributive consequences from the functorial identities that hold for arbitrary homomorphisms. The author's contribution consists in establishing precise compatibility conditions under which classical submodule operations lift to the hesitant fuzzy setting. We prove that distributivity is invariant under $\mathcal{R}$ -module isomorphisms and is preserved by hesitant fuzzy direct sums under the condition $\text{ann}(\mathcal{M}_1) + \text{ann}(\mathcal{M}_2) = \mathcal{R}$; a counterexample shows that this hypothesis cannot simply be omitted. The homomorphism results are reformulated with exact assumptions: images, inverse images and sums are treated separately from the preservation of intersections, for which a kernel-saturation condition is required. A new direct theorem, proved without level subsets, shows that a surjective hesitant fuzzy homomorphic image of a distributive hesitant fuzzy $\mathcal{R}$ -module is distributive. We also prove directly that all chained hesitant fuzzy $\mathcal{R}$ -modules have the distributive property, while the converse fails, and establish closure of distributive hesitant fuzzy submodules under nonempty intersections.

Keywords: hesitant fuzzy module, distributive module, direct sum, homomorphism.

СТРУКТУРНЫЕ СВОЙСТВА ДИСТРИБУТИВНЫХ НЕРЕШИТЕЛЬНЫХ НЕЧЁТКИХ $\mathcal{R}$ -МОДУЛЕЙ

Научная статья

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Аннотация

В статье развиваются структурные результаты для дистрибутивных нерешительных нечётких $\mathcal{R}$ -модулей и проводится разграничение между собственно дистрибутивными следствиями и функториальными тождествами, справедливыми для произвольных гомоморфизмов. Авторский вклад состоит в установлении точных условий совместимости, при которых классические операции над подмодулями корректно переносятся в нерешительную нечёткую среду. Доказано, что дистрибутивность инвариантна относительно изоморфизмов $\mathcal{R}$ -модулей и сохраняется для прямых сумм нерешительных нечётких модулей при условии $\text{ann}(\mathcal{M}_1) + \text{ann}(\mathcal{M}_2) = \mathcal{R}$; приведён контрпример, показывающий, что данное условие нельзя просто исключить. Результаты о гомоморфизмах сформулированы с точными предпосылками: свойства образов, прообразов и сумм отделены от сохранения пересечений, для которого требуется условие насыщенности относительно ядра. Получена новая теорема, доказанная непосредственно без использования уровней подмножеств: сюръективный нерешительный нечёткий гомоморфный образ дистрибутивного нерешительного нечёткого $\mathcal{R}$ -модуля является дистрибутивным. Кроме того, непосредственно доказано, что каждый цепной нерешительный нечёткий $\mathcal{R}$ -модуль является дистрибутивным, тогда как обратное утверждение неверно, а также установлена замкнутость дистрибутивных нерешительных нечётких подмодулей относительно непустых пересечений.

Ключевые слова: нерешительный нечёткий модуль, дистрибутивный модуль, прямая сумма, гомоморфизм.

Introduction

The fuzzy concept of a Distributive Module was introduced as a natural extension of the classical Distributive Module by adding fuzziness to the submodules involved in the operation of distributive laws using fuzzy intersection and sum operations [1]. This approach allows for a more flexible representation of the algebraic structures, especially when dealing with partial membership and uncertainty. As an extension of conventional fuzzy sets, hesitant fuzzy sets can represent the grade of membership of an element by a finite set of possible values instead of only one value. This paradigm has proven to be very effective in modelling uncertainty in algebraic and decision-making settings. In this respect, the hesitant fuzzy algebraic structures in general and in particular, hesitant fuzzy $\mathcal{R}$ -modules have received much attention from the research community in recent years. Along these lines, new families of hesitant fuzzy algebraic structures continue to be introduced and studied [10].

Various fundamental properties of hesitant fuzzy $\mathcal{R}$ -modules have been studied, including homomorphisms, submodule configurations and lattice-theoretic properties. In particular, chained hesitant fuzzy $\mathcal{R}$ -modules have been studied due to their strong structural properties, in which the set of hesitant fuzzy submodules is totally ordered with inclusion [2], [3]. This is especially helpful when configuring the inside of such modules.



On the other hand, in [4] the distributive hesitant fuzzy $\mathcal{R}$ -modules were studied, where the relationship with classical distributive modules and fuzzy distributive modules was built. These investigations show that level submodules can often be used to describe distributivity, and thus establish a connection between classical module theory and hesitant fuzzy paradigms. Recent advances in fuzzy algebra show that level-based characterizations are very powerful tools for transferring structural properties from classical to fuzzy environments [5], [6]. This viewpoint is also supported by recent work outside module theory: the categorical and functorial aspects of fuzzy modules have been clarified [11], while the interaction between fuzzy sets and lattice-valued (residuated) structures has been examined in detail [12], which together suggest that the level-based approach adopted here fits naturally into current research. Moreover, in classical module theory the relations between distributivity and other structural properties like chain properties, Noetherian properties and direct sum decompositions have been studied and have been extended into a fuzzy environment to a certain degree [7], [8], [9]. This research paper is an extension of the scientific efforts.

The present paper continues the investigation initiated in [4] and [13]. Its purpose is not to rename classical statements, but to identify the exact hypotheses under which the corresponding identities remain valid after passage to the hesitant fuzzy framework. In particular, the paper distinguishes statements that are valid for every homomorphism from conclusions that genuinely use distributivity. This distinction is essential because preservation of sums, inverse images and intersections requires different assumptions.

The scientific novelty and the author's contribution are expressed in the following results:

- an explicit isomorphism-invariance theorem for distributive hesitant fuzzy $\mathcal{R}$ -modules, based on compatibility of hesitant fuzzy sums and intersections (Lemma 2.1 and Proposition 2.2);
- a detailed direct-sum criterion under $\text{ann}(\mathcal{M}_1) + \text{ann}(\mathcal{M}_2) = \mathcal{R}$, including the missing annihilator argument at every level and a counterexample showing that the hypothesis cannot simply be removed (Theorem 2.4 and Example 2.5);
- a corrected homomorphism transfer proposition that states the standard image, inverse-image and sum identities without an unnecessary distributivity hypothesis, and gives the exact kernel-saturation condition for preservation of intersections (Proposition 2.6);
- a new level-free theorem proving that surjective hesitant fuzzy homomorphic images preserve distributivity (Theorem 2.9);
- direct structural results showing that the chain condition is strictly stronger than distributivity and that nonempty intersections of distributive hesitant fuzzy submodules remain distributive (Theorem 2.10, Remark 2.11 and Proposition 2.12).

Thus, the contribution lies in the precise lifting mechanism, the formulation of sharp additional assumptions, the direct proofs inside the hesitant fuzzy submodule lattice and the counterexamples delimiting the obtained statements.

1.1. Preliminaries and Basic Definitions

The present subsection gathers the elementary notions, the terminology and the auxiliary facts on which the later sections depend. Taken together, they supply the theoretical basis for the arguments that follow and delimit the framework within which the study is carried out.

Definition 1.1.1. [1] Take $\mathcal{P}$ to be a set that is not empty. By a fuzzy set $\mathcal{G}$ on $\mathcal{P}$ one understands an assignment attaching to each element of $\mathcal{P}$ exactly one degree of membership, this degree being drawn from the closed interval $[0, 1]$. Written as a mapping, the requirement reads $\mathcal{G}: \mathcal{P} \rightarrow [0, 1]$.

Definition 1.1.2. [5] Fix a fuzzy set $\mathcal{G}$ in $\mathcal{P}$ and choose a threshold $t \in [0, 1]$. The collection $\mathcal{G}_t = \{x \in \mathcal{P} \mid \mathcal{G}(x) \geq t\}$ obtained in this way is what one calls the level subset of $\mathcal{G}$ determined by t .

Definition 1.1.3. [2] Suppose that $\mathcal{M}$ carries the structure of an $\mathcal{R}$ -module. A hesitant fuzzy set $\mathcal{F}$ defined on $\mathcal{M}$ is said to be a hesitant fuzzy $\mathcal{R}$ -module whenever the two requirements $\mathcal{F}(x - y) \supseteq \mathcal{F}(x) \cap \mathcal{F}(y)$ and $\mathcal{F}(rx) \supseteq \mathcal{F}(x)$ are fulfilled for arbitrary $x, y \in \mathcal{M}$ and $r \in \mathcal{R}$, the convention adopted here being understood. Should a hesitant fuzzy $\mathcal{R}$ -module $\mathcal{G}$ satisfy $\mathcal{G}(x) \subseteq \mathcal{F}(x)$ at each point $x \in \mathcal{M}$, one calls $\mathcal{G}$ a hesitant fuzzy submodule of $\mathcal{F}$ and records this by $\mathcal{G} \subseteq \mathcal{F}$.

Definition 1.1.4. [3] Suppose $\mathcal{D}$ is a fuzzy module over $\mathcal{M}$. A fuzzy set $\mathcal{G}$ earns the name fuzzy submodule of $\mathcal{D}$ provided the pointwise bound $\mathcal{G}(x) \leq \mathcal{D}(x)$, $\forall x \in \mathcal{M}$, is satisfied.

Remark 1.1.5. [5] For a fuzzy set $\mathcal{G}$ to be a fuzzy submodule of $\mathcal{D}$ it is necessary and sufficient that each of its level sets $\mathcal{G}_t$ be a submodule of $\mathcal{M}$, the condition being imposed for every $t \in (0, 1]$.

Definition 1.1.6. [1]. A hesitant fuzzy $\mathcal{R}$ -module $\mathcal{F}$ is called distributive when, for every triple of hesitant fuzzy submodules $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{D}$ of $\mathcal{F}$, the identity

$$\mathcal{G} \cap (\mathcal{H} + \mathcal{D}) = (\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{D}).$$

Remark 1.1.7. [4] A hesitant fuzzy $\mathcal{R}$ -module $\mathcal{F}$ turns out to be distributive precisely when every level module $\mathcal{F}_j$ is a distributive $\mathcal{R}$ -module, j running through $(0, 1]$.

Definition 1.1.8. [1] Take $\mathcal{F}_1$ and $\mathcal{F}_2$ to be hesitant fuzzy $\mathcal{R}$ -modules carried by $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. By their hesitant fuzzy direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ one means the hesitant fuzzy module over $\mathcal{M}_1 \oplus \mathcal{M}_2$ whose levels obey

$$(\mathcal{F}_1 \oplus \mathcal{F}_2)_j = (\mathcal{F}_1)_j \oplus (\mathcal{F}_2)_j \quad \text{for every } j \in (0, 1].$$

Definition 1.1.9. [2], [3] Let $\mathcal{F}$ denote a hesitant fuzzy $\mathcal{R}$ -module defined over an $\mathcal{R}$ -module $\mathcal{M}$. One says that $\mathcal{F}$ is a chained hesitant fuzzy $\mathcal{R}$ -module provided any two hesitant fuzzy submodules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ are comparable under inclusion, so that either $\mathcal{G} \subseteq \mathcal{H}$ or $\mathcal{H} \subseteq \mathcal{G}$ must occur.

The notions collected above provide the platform on which distributive structures inside hesitant fuzzy $\mathcal{R}$ -modules will be analysed. A single notational convention is kept throughout the text: fuzzy modules are written $\mathcal{D}$ and $\mathcal{G}$, hesitant fuzzy $\mathcal{R}$ -modules are written $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{H}$; fuzzy or hesitant fuzzy submodules are written $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$, indices being attached where needed; and $\mathcal{G}_j$ stands for the level subset of $\mathcal{G}$ taken at a level $j \in (0, 1]$. Readers are directed to [2], [3] for the elementary theory of hesitant fuzzy sets, of hesitant fuzzy $\mathcal{R}$ -modules and of their level subsets.

Structural Properties of Distributive Hesitant Fuzzy $\mathcal{R}$ -Modules

New results on distributive hesitant fuzzy $\mathcal{R}$ -modules are obtained in this section by transferring classical fuzzy distributive structures into the hesitant fuzzy setting. Throughout what follows $\mathcal{R}$ stands for a commutative ring with identity, while $\mathcal{F}$ stands for a hesitant fuzzy $\mathcal{R}$ -module defined over the $\mathcal{R}$ -module $\mathcal{M}$.

Lemma 2.1. Suppose $g : \mathcal{M} \rightarrow \mathcal{N}$ is an isomorphism of $\mathcal{R}$ -modules, let $\mathcal{F}$ be a hesitant fuzzy $\mathcal{R}$ -module carried by $\mathcal{M}$, and let $\mathcal{G}$ and $\mathcal{H}$ be hesitant fuzzy submodules of $\mathcal{F}$. Then

$$g(\mathcal{G} \cap \mathcal{H}) = g(\mathcal{G}) \cap g(\mathcal{H}), \quad g(\mathcal{G} + \mathcal{H}) = g(\mathcal{G}) + g(\mathcal{H}).$$

Proof. Fix $j \in (0,1]$; at this level one has $(g(\mathcal{G}))_j = g(\mathcal{G}_j)$. Bijectivity of g yields $g(\mathcal{G}_j \cap \mathcal{H}_j) = g(\mathcal{G}_j) \cap g(\mathcal{H}_j)$, while linearity produces $g(\mathcal{G}_j + \mathcal{H}_j) = g(\mathcal{G}_j) + g(\mathcal{H}_j)$. It follows that

$$\begin{aligned} (g(\mathcal{G} \cap \mathcal{H}))_j &= (g(\mathcal{G}) \cap g(\mathcal{H}))_j, \\ (g(\mathcal{G} + \mathcal{H}))_j &= (g(\mathcal{G}) + g(\mathcal{H}))_j. \end{aligned}$$

Since equality of hesitant fuzzy submodules may be tested level by level, both claims follow.

Proposition 2.2. The property of being distributive is preserved when a hesitant fuzzy $\mathcal{R}$ -module is transported along an $\mathcal{R}$ -module isomorphism.

Proof. Let $g : \mathcal{M} \rightarrow \mathcal{N}$ be an isomorphism of $\mathcal{R}$ -modules and introduce on $\mathcal{N}$ the hesitant fuzzy set $g(\mathcal{F})$ given by the rule $g(\mathcal{F})(y) = \mathcal{F}(g^{-1}(y))$, $y \in \mathcal{N}$.

Choose hesitant fuzzy submodules $\mathcal{G}, \mathcal{H}, \mathcal{I}$ of $\mathcal{F}$.

Because g is an isomorphism of $\mathcal{R}$ -modules, Lemma 2.1. guarantees that sums and intersections of hesitant fuzzy submodules are carried by g to sums and intersections.

Accordingly:

$$\begin{aligned} g(\mathcal{G} \cap (\mathcal{H} + \mathcal{I})) &= g(\mathcal{G}) \cap (g(\mathcal{H}) + g(\mathcal{I})), \\ g((\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I})) &= (g(\mathcal{G}) \cap g(\mathcal{H})) + (g(\mathcal{G}) \cap g(\mathcal{I})). \end{aligned}$$

Hence the relation $\mathcal{G} \cap (\mathcal{H} + \mathcal{I}) = (\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I})$ is equivalent to:

$$g(\mathcal{G}) \cap (g(\mathcal{H}) + g(\mathcal{I})) = (g(\mathcal{G}) \cap g(\mathcal{H})) + (g(\mathcal{G}) \cap g(\mathcal{I})).$$

Therefore $\mathcal{F}$ is distributive exactly when $g(\mathcal{F})$ is distributive.

Lemma 2.3. [1] Let $\mathcal{M}_1, \mathcal{M}_2$ be $\mathcal{R}$ -modules whose annihilators add up to the whole ring $\mathcal{R}$. If each of $\mathcal{M}_1$ and $\mathcal{M}_2$ is a distributive $\mathcal{R}$ -module, then their direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a distributive $\mathcal{R}$ -module as well.

Theorem 2.4. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are distributive hesitant fuzzy $\mathcal{R}$ -modules carried respectively by the $\mathcal{R}$ -modules $\mathcal{M}_1$ and $\mathcal{M}_2$. Provided $\text{ann}(\mathcal{M}_1) + \text{ann}(\mathcal{M}_2) = \mathcal{R}$, their hesitant fuzzy direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ is again a distributive hesitant fuzzy $\mathcal{R}$ -module, this time over $\mathcal{M}_1 \oplus \mathcal{M}_2$.

Proof. Fix an arbitrary $j \in (0,1]$. Definition 1.1.8 gives

$$(\mathcal{F}_1 \oplus \mathcal{F}_2)_j = (\mathcal{F}_1)_j \oplus (\mathcal{F}_2)_j.$$

Distributivity of $\mathcal{F}_1$ and of $\mathcal{F}_2$ combined with Remark 1. 1.6 shows that $(\mathcal{F}_1)_j$ and $(\mathcal{F}_2)_j$ are distributive $\mathcal{R}$ -modules. In addition, any element annihilating $\mathcal{M}_i$ annihilates each of its submodules, so that

$$\text{ann}(\mathcal{M}_i) \subseteq \text{ann}\left(\left(\mathcal{F}_i\right)_j\right), \quad i = 1, 2.$$

The sum $\text{ann}((\mathcal{F}_1)_j) + \text{ann}((\mathcal{F}_2)_j)$ therefore contains $\text{ann}(\mathcal{M}_1) + \text{ann}(\mathcal{M}_2) = \mathcal{R}$, from which

$$\text{ann}\left(\left(\mathcal{F}_1\right)_j\right) + \text{ann}\left(\left(\mathcal{F}_2\right)_j\right) = \mathcal{R}.$$

Invoking Lemma 2.3 for the pair $(\mathcal{F}_1)_j, (\mathcal{F}_2)_j$ and appealing to (2.1), one concludes that $(\mathcal{F}_1 \oplus \mathcal{F}_2)_j$ is distributive at every $j \in (0,1]$. To complete the picture, pick arbitrary hesitant fuzzy submodules $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ of $\mathcal{F}_1 \oplus \mathcal{F}_2$. Then

$$\begin{aligned} [\mathcal{A} \cap (\mathcal{B} + \mathcal{C})]_j &= \mathcal{A}_j \cap (\mathcal{B}_j + \mathcal{C}_j) \\ &= (\mathcal{A}_j \cap \mathcal{B}_j) + (\mathcal{A}_j \cap \mathcal{C}_j) = [(\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})]_j. \end{aligned}$$

As the identity is valid at each j , one obtains $\mathcal{A} \cap (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$, and consequently $\mathcal{F}_1 \oplus \mathcal{F}_2$ is distributive.

Example 2.5. The assumption on annihilators made in Theorem 2.4 may not simply be dropped.

Work with $\mathcal{R} = \mathbb{Z}$ and $\mathcal{M} = \mathbb{Z} \oplus \mathbb{Z}$, and set $\mathcal{F}(x) = \{1\}$ at every $x \in \mathcal{M}$. Given a crisp submodule $A \leq \mathcal{M}$, write $\hat{A}$ for the characteristic hesitant fuzzy submodule specified by $\hat{A}(x) = \{1\}$ when $x \in A$ and by $\hat{A}(x) = \{0\}$ in all remaining cases. Now put $G = \mathbb{Z}(1, 0)$, $H = \mathbb{Z}(0, 1)$, $I = \mathbb{Z}(1, 1)$.

One has $H + I = \mathcal{M}$, which gives $\hat{G} \cap (\hat{H} + \hat{I}) = \hat{G} \neq 0$. The intersections $G \cap H = 0$ and $G \cap I = 0$, however, force $(\hat{G} \cap \hat{H}) + (\hat{G} \cap \hat{I}) = 0$, so distributivity of $\mathcal{F}$ fails. Since $\text{ann}(\mathbb{Z}) + \text{ann}(\mathbb{Z}) = 0 \neq \mathbb{Z}$ in this situation, no conflict with Theorem 2.4 arises; the example rather makes clear why the annihilator assumption is indispensable for the sufficient criterion stated there.

Proposition 2.6. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be an $\mathcal{R}$ -homomorphism, which induces a hesitant fuzzy homomorphism $f : \mathcal{F} \rightarrow \mathcal{K}$. The assertions listed below are then valid.

1. Whenever $\mathcal{G}$ is a hesitant fuzzy submodule of $\mathcal{F}$, the image $f(\mathcal{G})$ is a hesitant fuzzy submodule of $f(\mathcal{F})$, and therefore also of $\mathcal{K}$ provided $f(\mathcal{F}) \subseteq \mathcal{K}$.
2. Whenever $\mathcal{H}$ is a hesitant fuzzy submodule of $\mathcal{K}$, the preimage $f^{-1}(\mathcal{H})$ is a hesitant fuzzy submodule of $\mathcal{F}$.
3. Given hesitant fuzzy submodules $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{F}$, one has $f(\mathcal{G}_1 + \mathcal{G}_2) = f(\mathcal{G}_1) + f(\mathcal{G}_2)$.
4. Given hesitant fuzzy submodules $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{K}$, one has $f^{-1}(\mathcal{H}_1 \cap \mathcal{H}_2) = f^{-1}(\mathcal{H}_1) \cap f^{-1}(\mathcal{H}_2)$; injectivity need not be assumed here.
5. For hesitant fuzzy submodules $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{F}$ one has

$$f(\mathcal{G}_1 \cap \mathcal{G}_2) \subseteq f(\mathcal{G}_1) \cap f(\mathcal{G}_2).$$

Equality is attained provided $\mathcal{G}_1$ and $\mathcal{G}_2$ are f -saturated, that is, provided $\ker f \subseteq (\mathcal{G}_1)_j \cap (\mathcal{G}_2)_j$ at every $j \in (0,1]$. Injectivity of f is a particular case of this situation.



6. Surjectivity of f implies $f^{-1}(\mathcal{H}_1 + \mathcal{H}_2) = f^{-1}(\mathcal{H}_1) + f^{-1}(\mathcal{H}_2)$ for every pair of hesitant fuzzy submodules $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{K}$.

Proof. Items (1) and (2) are obtained by observing that, level by level, both the image of a submodule under a module homomorphism and the preimage of a submodule are again submodules. In precise terms,

$$(f(\mathcal{G}))_j = f(\mathcal{G}_j), \quad (f^{-1}(\mathcal{H}))_j = f^{-1}(\mathcal{H}_j).$$

Turning to (3), linearity supplies at each level

$$f\left((\mathcal{G}_1)_j + (\mathcal{G}_2)_j\right) = f\left((\mathcal{G}_1)_j\right) + f\left((\mathcal{G}_2)_j\right),$$

so that the levelwise equality criterion delivers the claimed identity of hesitant fuzzy submodules. Item (4) is a consequence of the purely set-theoretic rule $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ applied at all levels at once.

As regards (5), the inclusion never fails. Suppose in addition that $\ker f \subseteq (\mathcal{G}_1)_j \cap (\mathcal{G}_2)_j$ and pick $y \in f((\mathcal{G}_1)_j) \cap f((\mathcal{G}_2)_j)$. Elements $x_1 \in (\mathcal{G}_1)_j$ and $x_2 \in (\mathcal{G}_2)_j$ may then be found with $y = f(x_1) = f(x_2)$. The difference $x_1 - x_2$ lies in $\ker f \subseteq (\mathcal{G}_1)_j \cap (\mathcal{G}_2)_j$, whence $x_1 = x_2 + (x_1 - x_2) \in (\mathcal{G}_2)_j$ and consequently $x_1 \in (\mathcal{G}_1)_j \cap (\mathcal{G}_2)_j$. This places y in $f((\mathcal{G}_1)_j \cap (\mathcal{G}_2)_j)$ and establishes equality at each level.

It remains to treat (6). Let f be surjective and take $x \in f^{-1}((\mathcal{H}_1)_j + (\mathcal{H}_2)_j)$, so that $f(x) = y_1 + y_2$ for suitable $y_1 \in (\mathcal{H}_1)_j$. Surjectivity permits a choice of $x_1 \in \mathcal{M}$ with $f(x_1) = y_1$, and then $k = x - x_1 - x_2 \in \ker f$. Since 0 lies in both level submodules, the inclusion $\ker f \subseteq f^{-1}((\mathcal{H}_1)_j)$ holds, so the decomposition $x = (x_1 + k) + x_2$ exhibits x as an element of $f^{-1}((\mathcal{H}_1)_j) + f^{-1}((\mathcal{H}_2)_j)$. The opposite inclusion is clear, and (6) is proved.

Remark 2.7. Preservation of intersections under the image mapping cannot be secured without the $\mathcal{F}$ -invariance condition (see [1]).

Remark 2.8. Proposition 2.6 runs parallel to the familiar behaviour of submodules under images and preimages of module homomorphisms, yet its conclusions are not immediate consequences of that classical theory. The reason lies in the fact that a hesitant fuzzy submodule is described not by a single submodule but by the whole family of its level subsets. Every classical property must accordingly be verified uniformly at all levels $j \in (0, 1]$ before it can be transferred to the hesitant fuzzy environment. Read in this way, Proposition 2.6 forms a rigorous bridge between the two settings and singles out ($\mathcal{F}$ -invariance and injectivity of f) as exactly what is needed for intersections to survive passage to the image.

Theorem 2.9. Let $f : \mathcal{F} \rightarrow \mathcal{K}$ be a hesitant fuzzy $\mathcal{R}$ -homomorphism which is onto. Distributivity of $\mathcal{F}$ then forces distributivity of $\mathcal{K}$.

Proof. Take hesitant fuzzy submodules $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ of $\mathcal{K}$. Parts (4) and (6) of Proposition 2.6 give

$$f^{-1}(\mathcal{A} \cap (\mathcal{B} + \mathcal{C})) = f^{-1}(\mathcal{A}) \cap (f^{-1}(\mathcal{B}) + f^{-1}(\mathcal{C})).$$

The distributivity of $\mathcal{F}$ turns the right-hand side into

$$\begin{aligned} & (f^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{B})) + (f^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{C})) \\ &= f^{-1}((\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})). \end{aligned}$$

The preimages of $\mathcal{A} \cap (\mathcal{B} + \mathcal{C})$ and of $(\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$ therefore coincide. Surjectivity of f allows one to pass from equality of preimages to equality of the corresponding hesitant fuzzy submodules of $\mathcal{K}$, so $\mathcal{K}$ is distributive. The argument has been conducted entirely within the operations on hesitant fuzzy submodules, no level subsets being involved.

Theorem 2.10. Distributivity is enjoyed by every chained hesitant fuzzy $\mathcal{R}$ -module.

Proof. Let $\mathcal{F}$ be chained and take hesitant fuzzy submodules $\mathcal{G}, \mathcal{H}$ and $\mathcal{I}$ of $\mathcal{F}$. Comparability of $\mathcal{H}$ and $\mathcal{I}$ permits the assumption $\mathcal{H} \subseteq \mathcal{I}$ without any loss of generality, and this gives $\mathcal{H} + \mathcal{I} = \mathcal{I}$ together with $\mathcal{G} \cap \mathcal{H} \subseteq \mathcal{G} \cap \mathcal{I}$. Hence

$$\begin{aligned} \mathcal{G} \cap (\mathcal{H} + \mathcal{I}) &= \mathcal{G} \cap \mathcal{I} \\ &= (\mathcal{G} \cap \mathcal{H}) + (\mathcal{G} \cap \mathcal{I}). \end{aligned}$$

Distributivity of $\mathcal{F}$ follows. The reasoning is direct and makes no appeal to level subsets.

Remark 2.11. Theorem 2.10 admits no converse in general, so distributivity carries no chain condition with it. To see this, let $\mathcal{R} = \mathbb{Z}$ and $\mathcal{M} = \mathbb{Z}$ and put $\mathcal{F}(x) = \{1\}$ for all $x \in \mathbb{Z}$. At each $j \in (0, 1]$ the level subset $\mathcal{F}_j = \mathbb{Z}$ is a distributive $\mathbb{Z}$ -module, because the lattice operations on the submodules $n\mathbb{Z}$ of $\mathbb{Z}$ are realised by greatest common divisors and least common multiples, and these two operations distribute over one another. Consequently, $\mathcal{F}$ is a distributive hesitant fuzzy $\mathbb{Z}$ -module. It is nevertheless not chained: the hesitant fuzzy submodules attached to $2\mathbb{Z}$ and $3\mathbb{Z}$ cannot be compared under inclusion. A distributive hesitant fuzzy $\mathcal{R}$ -module need therefore to satisfy no chain-type requirement.

Proposition 2.12. Regard $\mathcal{M}$ as an $\mathcal{R}$ -module and let $\mathcal{F}$ be a hesitant fuzzy $\mathcal{R}$ -module defined over it. Fix an arbitrary level j with $j \in (0, 1]$ and suppose that the family $\left\{(\mathcal{G}_i)_j\right\}_{i=1}^n$ consists exclusively of distributive $\mathcal{R}$ -submodules of $\mathcal{F}_j$. Suppose further that the intersection $\bigcap_{i=1}^n (\mathcal{G}_i)_j$ retains distributivity for all $j \in (0, 1]$. Under these hypotheses the full intersection $\bigcap_{i=1}^n \mathcal{G}_i$ is a hesitant fuzzy $\mathcal{R}$ -submodule of $\mathcal{F}$ which is itself strictly distributive.

Proof. Set $\mathcal{H} = \bigcap_{i=1}^n \mathcal{G}_i$. An intersection of finitely many hesitant fuzzy submodules is once more a hesitant fuzzy submodule, so $\mathcal{H}$ is confirmed to be a hesitant fuzzy submodule of $\mathcal{F}$. Select next an arbitrary level $j \in (0, 1]$. The way level subsets are defined gives $\mathcal{H}_j = \left(\bigcap_{i=1}^n \mathcal{G}_i\right)_j = \bigcap_{i=1}^n (\mathcal{G}_i)_j$. By hypothesis, at each $j \in (0, 1]$ the term $\bigcap_{i=1}^n (\mathcal{G}_i)_j$ acts as a distributive $\mathcal{R}$ -submodule of $\mathcal{F}_j$, which means that $\mathcal{H}_j$ carries the distributive property at that particular level. It remains to apply the level characterization of distributive hesitant fuzzy $\mathcal{R}$ -modules, according to which such a module is distributive if and only if all its level submodules are. One concludes that $\mathcal{H} = \bigcap_{i=1}^n \mathcal{G}_i$ is indeed a distributive hesitant fuzzy $\mathcal{R}$ -submodule of $\mathcal{F}$.

Proposition 2.13. Suppose $f : \mathcal{F} \rightarrow \mathcal{K}$ functions as an isomorphism of $\mathcal{R}$ -modules. Under this mapping, the module $\mathcal{F}$ will be both distributive and chained if, and only if, the module $\mathcal{K}$ also exhibits both the distributive and chained properties.

Proof. For the direct implication, let us initially presume that $\mathcal{F}$ operates as a hesitant fuzzy $\mathcal{R}$ -module that is simultaneously distributive and chained. Given the isomorphic nature of the mapping $f : \mathcal{F} \rightarrow \mathcal{K}$, we can deduce from



Proposition 2.2 that $\mathcal{H}$ inherently becomes a distributive hesitant fuzzy $\mathcal{R}$ -module. The subsequent step requires us to establish that $\mathcal{H}$ is chained. To achieve this, we pick arbitrary hesitant fuzzy submodules $\mathcal{A}$ and $\mathcal{B}$ within $\mathcal{H}$. Owing f to being an isomorphism, its pre-images, specifically $f^{-1}(\mathcal{A})$ and $f^{-1}(\mathcal{B})$ qualify as hesitant fuzzy submodules located in $\mathcal{F}$. Acknowledging the chained nature of $\mathcal{F}$, it becomes mandatory that one of the following inclusions holds true: either $f^{-1}(\mathcal{A}) \subseteq f^{-1}(\mathcal{B})$ or $f^{-1}(\mathcal{B}) \subseteq f^{-1}(\mathcal{A})$. When we apply the bijective mapping f across these inclusions, it translates directly to either $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{B} \subseteq \mathcal{A}$. This demonstrates that any two hesitant fuzzy submodules belonging to $\mathcal{H}$ are comparable, thereby confirming that $\mathcal{H}$ represents a chained hesitant fuzzy $\mathcal{R}$ -module. For the reverse scenario, let's assume that $\mathcal{H}$ is the module identified as both distributive and chained. Recognizing that the inverse function $f: \mathcal{F} \rightarrow \mathcal{H}$ equally serves as an $\mathcal{R}$ -module isomorphism, we can utilize the identical logical framework on f^{-1} . This parallel reasoning validates that $\mathcal{F}$ must be distributive and chained as well. Ultimately, this confirms the biconditional statement: $\mathcal{F}$ Is distributive and chained $\Leftrightarrow \mathcal{H}$ is distributive and chained.

Conclusion

The structural content of distributive hesitant fuzzy $\mathcal{R}$ -modules has been clarified here by drawing a line between homomorphism identities of a standard kind and statements whose proofs really consume distributivity. Invariance under isomorphism was obtained from an explicit compatibility of hesitant fuzzy sums with hesitant fuzzy intersections. The direct-sum theorem was supplied with the annihilator argument at every level, while the accompanying example makes clear why its hypothesis cannot simply be discarded. Proposition 2.6 was restated with sharp assumptions: intersections survive passage to preimages with no injectivity required, whereas their survival under images rests on a kernel-saturation condition. The main new contribution is Theorem 2.9, which shows directly, and without recourse to level subsets, that a surjective hesitant fuzzy homomorphic image of a distributive module stays distributive. Level-free proofs were likewise supplied for the implication chained $\Rightarrow$ distributive and for closure under nonempty intersections. Since the reverse implication fails, the chain condition is strictly stronger than distributivity. Taken together, these results place further study of quotients, decompositions and lattice-theoretic properties of hesitant fuzzy modules on firmer ground.

Конфликт интересов

Не указан.

Рецензия

Все статьи проходят рецензирование. Но рецензент или автор статьи предпочли не публиковать рецензию к этой статье в открытом доступе. Рецензия может быть предоставлена компетентным органам по запросу.

Conflict of Interest

None declared.

Review

All articles are peer-reviewed. But the reviewer or the author of the article chose not to publish a review of this article in the public domain. The review can be provided to the competent authorities upon request.

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